Introduction to Mathematical Quantum Theory

Text of the Exercises

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Exercise 1

a Consider the function $f \in L^1(\mathbb{T})$ defined as the periodization of

$$f(x) := x(2\pi - x). \tag{1}$$

Calculate the Fourier coefficients of f and use them to prove that

$$\sum_{k=0}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{2}$$

b Let σ be a positive real number and $\mathbf{v},\ \mathbf{u} \in \mathbb{R}^d$. Consider the function $g_{\sigma,\mathbf{v},\mathbf{u}}$ in the space $L^2\left(\mathbb{R}^d\right)$ with $d \in \mathbb{N}$ defined as

$$g_{\sigma,\mathbf{v},\mathbf{u}}(\mathbf{x}) := \left(\frac{\sigma}{\pi}\right)^{\frac{d}{4}} e^{-\frac{\sigma}{2}|\mathbf{x}-\mathbf{v}|^2 + i\mathbf{u}\cdot\mathbf{x}}. \tag{3}$$

Then prove that $\widehat{g}_{\sigma,\mathbf{v},\mathbf{u}} = e^{i\mathbf{v}\cdot\mathbf{u}} g_{\sigma^{-1},\mathbf{u},-\mathbf{v}}$, i.e.

$$\mathcal{F}\left[\left(\frac{\sigma}{\pi}\right)^{\frac{d}{4}} e^{-\frac{\sigma}{2}|\mathbf{x}-\mathbf{v}|^2 + i\mathbf{u}\cdot\mathbf{x}}\right](\mathbf{k}) = \left(\frac{1}{\sigma\pi}\right)^{\frac{d}{4}} e^{-\frac{1}{2\sigma}|\mathbf{k}-\mathbf{u}|^2 - i\mathbf{u}\cdot(\mathbf{k}-\mathbf{v})}. \tag{4}$$

Exercise 2

Consider V_1 and V_2 two normed vector spaces over[1] $\mathbb{F}$ and $T : V_1 \to V_2$ a linear mapping. Define $\|T\|_{V_1,V_2}$ as

$$\|T\| := \sup_{v\in V_1,\ v\neq 0} \frac{\|Tv\|}{\|v\|}. \tag{5}$$

For a generic linear mapping T we have $\|T\| \in [0, +\infty]$. Prove that

$$\|T\| = \sup_{v\in V_1,\ \|v\|_{V_1}=1} \|Tv\| \tag{6}$$

$$= \sup_{v\in V_1,\ \|v\|_{V_1}\leqslant 1} \|Tv\|. \tag{7}$$

Prove moreover that the following are equivalent

a T is continuous.

b T is continuous in 0, meaning that for any sequence $\{v_n\}_{n\in\mathbb{N}} \subseteq V_1$,

$$v_n \to 0 \implies Tx_n \to 0. \tag{8}$$

c The quantity $\|T\|$ is finite, meaning that $\|T\| < +\infty$.

[1] Here and in the following $\mathbb{F}$ can be chosen to be either $\mathbb{R}$ or $\mathbb{C}$.

Exercise 3 (Young Inequality)

Consider $p,\ q,\ r \in [1, +\infty]$ such that

$$\frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{p}. \tag{9}$$

Let $f \in L^q\left(\mathbb{R}^d\right)$, $g \in L^r\left(\mathbb{R}^d\right)$; prove that

$$\|f * g\|_p \leqslant \|f\|_q \|g\|_r . \tag{10}$$

Hint: Consider the functions $\alpha,\ \beta,\ \gamma$ *defined as*

$$\alpha(\mathbf{x}, \mathbf{y}) := |f(\mathbf{y})|^q |g(\mathbf{x} - \mathbf{y})|^r , \tag{11}$$

$$\beta(\mathbf{y}) := |f(\mathbf{y})|^q , \tag{12}$$

$$\gamma(\mathbf{x}, \mathbf{y}) := |g(\mathbf{x} - \mathbf{y})|^r , \tag{13}$$

notice that

$$|f * g(\mathbf{x})| \leqslant \int_{\mathbb{R}^d} \alpha(\mathbf{x}, \mathbf{y})^{\frac{1}{p}} \beta(\mathbf{y})^{\frac{p-q}{pq}} \gamma(\mathbf{x}, \mathbf{y})^{\frac{p-r}{pr}} d\mathbf{y} \tag{14}$$

and that

$$\frac{1}{p} + \frac{p-q}{pq} + \frac{p-r}{pr} = 1 \tag{15}$$

to apply Hölder inequality.

Exercise 4

a Prove that there exists a positive real number C such that we have

$$\sup_{0 \leqslant a < b < +\infty} \left| \int_a^b \frac{\sin x}{x} dx \right| \leqslant C. \tag{16}$$

Hint: Consider the function

$$F(t) := \int_0^\eta e^{-tx} \frac{\sin x}{x} dx. \tag{17}$$

Deduce a bound on $F'(t)$ *uniform in* η*. Apply the fundamental theorem of calculus for* $F(0)$ *to conclude.*

b Consider an odd function $f \in L^1(\mathbb{R})$. Prove that for any such function we have

$$\sup_{0 \leqslant a < b < +\infty} \left| \int_a^b \frac{\widehat{f}(k)}{k} dk \right| \leqslant \frac{C}{(2\pi)^{\frac{d}{2}}} \|f\|_1 . \tag{18}$$

c Let $g(k)$ be a continuous odd function on the line such that is equal to $1/\log k$ for any $k \geqslant 2$. Prove that there cannot be an $L^1(\mathbb{R})$ function whose Fourier transform is g.